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FORCED PERIODIC MOTIONS IN NON-HAMILTONIAN SYSTEMS WITH ONE DEGREE OF FREEDOM

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The method of successive approximations is used for deriving approximations of the basic resonance solution of the nonlinear system with one degree of freedom, which is more general than those considered in [1-3], without reduction to a system with a rotating phase [4]. Such approach is of methodological interest and has the advantage of clear physical meaning. The stability of perturbed motion is analyzed by the first Liapunov method [5]. The theorems of existence and stability of the resonant solution are of a constructive kind, and the sufficient conditions are obtained in a compact and readily understood form, and in a number of cases are explicitly obtained with the use of integrals of the unperturbed equation.
An important particular case of a system with one degree of freedom is considered, and a specific example of calculations is presented.

1. Statement of the problem. A system with one degree of freedom defined by the equation

$$x^{\cdot\cdot} + Q(x, x^{\cdot}) = \varepsilon q(t, x, x^{\cdot}, \varepsilon) \tag{1.1}$$

where x is the generalized coordinate $x^{\cdot} = dx / dt$ is the velocity, and $\varepsilon \geqslant 0$ is a small parameter, is considered.

It is assumed that the unperturbed equation

$$x_0^{\cdot\cdot} + Q(x_0, x_0^{\cdot}) = 0 \tag{1.2}$$

admits a complete set of periodic solutions to which appertain: a) oscillating motions $x_0 = \varphi(\psi, a)$, $x_0^{\cdot} = \omega \varphi_\psi'(\psi, a)$; and b) rotary motions $x_0 = \psi + \eta(\psi, a)$, $x_0^{\cdot} = \omega[1 + \eta_\psi'(\psi, a)]$. In these formulas φ and η are some periodicfunctions of phase $\psi = \omega(a)(t + \tau)$ of 2π period and of the constant of integration a (a is the first integral of Eq. (1.2) τ is the phase constant, and $\omega(a)$ is the natural frequency of oscillations or rotations).

It follows from (1.2) that for the considered type of motion the "work of internal forces" during the period $T_0 = 2\pi / \omega$ is zero

$$\int_0^{T_0} Q(x_0, x_0^{\cdot}) x_0^{\cdot}\, dt = \oint Q(x_0, x_0^{\cdot})\, dx_0 \equiv 0 \tag{1.3}$$

In the case of rotational motions it is, furthermore, necessary that function Q be 2π-periodic in $x : Q(x + 2\pi, x^{\cdot}) \equiv Q(x, x^{\cdot})$. The identity (1.3) is then of a simpler form. Let us assume that the first integral of Eq. (1.2), which is solvable for $x_0^{\cdot}(x_0, a)$, is known and that $x_0^{\cdot}(x_0 + 2\pi, a) \equiv x_0^{\cdot}(x_0, a)$. Function $x_0^{\cdot}$ is determined as the general 2π-periodic solution of constant sign of the phase trajectory equation

$$dx_0^{\cdot} / dx_0 = Q(x_0, x_0^{\cdot}) / x_0^{\cdot} \tag{1.4}$$

In that case formula (1.3) must be satisfied independently of a

$$\int_0^{2\pi} Q(x, x_0^{\cdot}(x, a))\, dx \equiv 0 \tag{1.5}$$

The following requirements are assumed to be satisfied relative to the perturbed system.

1. In addition to the indicated above properties function $Q(x, x^{\cdot})$ has second partial derivatives that satisfy the Lipschitz conditions. In the case of oscillations the range of argument variation is bounded, while in that of rotation it is sufficient to consider the interval $[0, 2\pi]$ owing to the periodicity of Q in x.

2. Function $q(t, x, x^{\cdot}, \varepsilon)$ is continuous and periodic in t of constant period $2\pi / \nu$, 2π-periodic in x in the case of rotation and admits first partial derivatives with respect to x, $x^{\cdot}$ and ε which satisfy the Lipschitz conditions with constants that are independent of t in the indicated region of variation of x, $x^{\cdot}$ and $\varepsilon \in [0, \varepsilon_0]$.

We have the problem of determining for the perturbed system (1.1) the resonance motions of period $T = 2\pi m / \nu = nT_0$, where m and n are relatively prime integers. Parameter a must then satisfy the equation

$$m\omega(a) = n\nu \tag{1.6}$$

and let a^* be a simple admissible root of (1.6). During time T the rotating variable x obtains a increment equal to $2\pi n$. The periodicity condition of the kind (1.3) is for the perturbed motion of the form

$$\int_0^{T} Q(x, x^{\cdot}) x^{\cdot}\, dt = \varepsilon \int_0^{T} q(t, x, x^{\cdot}, \varepsilon) x^{\cdot}\, dt, \quad x = x(t, \varepsilon) \tag{1.7}$$

and must be satisfied for any $\varepsilon \in [0, \varepsilon_0]$.

In the considered here problem the initial instant of time is immaterial, since the method separates individual periodic motions. Hence it is possible to set $t_0 = 0$, find the related stationary solution, and, then, determine x and $x^{\cdot}$ that correspond to some initial instant t_0. The specification of such initial conditions ensures that the motion is along a stationary trajectory.

2. Derivation of the perturbed resonance solution. The substitution $x(t, \varepsilon) = x_0(\psi, a^*) + \varepsilon y(t, \varepsilon)$, where y is the unknown function of t of period T, reduces Eq. (1.1) to the quasi-linear form

$$y^{\cdot\cdot} + (Q_{x}') y^{\cdot} + (Q_x') y = (q) + \varepsilon R(t, y, y^{\cdot}, \varepsilon) \qquad (2.1)$$
$$R = -{}^1/_2 (Q_{\dot{x}\cdot}'') y^{\cdot 2} - {}^1/_2 (Q_{x}'') y^2 - (Q_{xx^{\cdot}}'') yy^{\cdot} + (q_{x^{\cdot}}') y^{\cdot} + (q_x') y + (q_\varepsilon') + r$$

where r is a periodic function of t which satisfies the Lipschitz conditions for $y, y^{\cdot}$, and ε, with constants independent of t, and vanishes when $\varepsilon = 0$. Expressions of the kind $(Q_{x^{\cdot}}'), (q), \ldots$ indicate that the argument of functions $x = x_0$, $x^{\cdot} = x_0^{\cdot}$, and $\varepsilon = 0$.

The quasi-linear equation (2.1) with periodic coefficients is solved by successive approximations with respect to the small parameter ε with the use of the method of varying the constants of integration

$$y_{i+1} = \alpha_{i+1} u + u \int_0^t dt_1 \left\{ \int_0^{t_1} [(q) + \varepsilon R_i] \frac{u}{\Delta} dt_2 - [(q) + \varepsilon R_i] \frac{v}{\Delta} + \beta_{i+1} \right\} + v \left\{ \int_0^t [(q) + \varepsilon R_i] \frac{u}{\Delta} dt_1 + \beta_{i+1} \right\}, \quad i = 0, 1, \ldots \qquad (2.2)$$

where α_{i+1} and β_{i+1} are constants chosen at each step so as to have y_{i+1} in the form of a periodic function; u and v are periodic functions of t, and Δ is the Wronskian. Their derivation is given below. Function R_i is determined in the preceding step to within constant α_i

$$R_i \equiv R(t, y_i, y_i^{\cdot}, \varepsilon) \qquad (2.3)$$

It is possible to assume without loss of accuracy that in the $(i+1)$-st step in function $r, y = y_{i-1}$ and $y^{\cdot} = \dot{y}_{i-1}$.

In the zero approximation function y is the periodic solution of the linear equation obtained from (2.1) with $\varepsilon = 0$. Its general solution is obtained from fundamental system of solution of the homogeneous equation

$$z^{\cdot\cdot} + (Q_{x^{\cdot}}') z^{\cdot} + (Q_x') z = 0 \qquad (2.4)$$

The linearly independent solutions of Eq. (2.4) are obtained by differentiating x_0 with respect to τ and a with $a = a^*$

$$z_1 = u = x_0^{\cdot}(\psi, a^*), \quad z_2 = ut + v \qquad (2.5)$$

The periodic function v is defined by the derivative of x_0 with respect to a : $v = [(\partial x_0 / \partial a) / (\ln \omega)']^*$ and is the periodic solution of the nonhomogeneous equation

$$v^{\cdot\cdot} + (Q_{x^{\cdot}}') v^{\cdot} + (Q_x') v = -2u^{\cdot} - (Q_{x^{\cdot}}') u \qquad (2.6)$$

Since a^* is a simple root of (1.6), hence $(\ln \omega)'^* \neq 0$. Function z_2 is a linear combination of z_1 and $z_3 = (Dx_0 / Da)^*$, where z_3 is also a solution of (2.4) and D is the symbol of a total partial derivative.

Because of the linear independence of z_1 and z_2 the Wronskian is nonzero for all t and on the strength of the Liouville theorem it is defined by

$$\Delta(t+\tau) = \begin{vmatrix} z_1 & z_2 \\ z_1^{\cdot} & z_2^{\cdot} \end{vmatrix} = u^2 + uv^{\cdot} - u^{\cdot}v = \Delta_0 \exp\left[-\int (Q_{x^{\cdot}}')\, dt\right] \qquad (2.7)$$

$$\Delta_0 = \Delta(0), \quad \Delta^{\cdot} = -(Q_{x^{\cdot}}')\Delta, \quad (1/\Delta)^{\cdot} = (Q_{x^{\cdot}}')(1/\Delta)$$

It follows from (2.7) that function $(Q_{x^{\cdot}}')$ has a zero mean with respect to t.

Using the method of variation of integration constants we can represent the sought general solution of y_0 in the form

$$y_0 = \alpha_0 u + u \int_0^t dt_1 \left[\int_0^{t_1} (q) \frac{u}{\Delta}\, dt_2 - (q)\frac{v}{\Delta} + \beta_0\right] + v\left[\int_0^t (q)\frac{u}{\Delta}\, dt_1 + \beta_0\right] \qquad (2.8)$$

It follows from (2.8) that function y_0 is periodic for any α_0 when condition

$$P(\tau) \equiv \int_0^T q(t, x_0, x_0^{\cdot}, 0)\frac{u}{\Delta}\, dt = 0 \qquad (2.9)$$

is satisfied and if we state

$$\beta_0 = -\frac{1}{T}\int_0^T dt\left[\int_0^t (q)\frac{u}{\Delta}\, dt_1 - (q)\frac{v}{\Delta}\right] \qquad (2.10)$$

If Eq. (2.9) has a real root τ^*, then the periodic solution of y_0 is determined to within the parameter α_0. The substitution of (2.10) and $\tau = \tau^*$ into (2.8) yields the expression $y_0 = \alpha_0 u + y_0^*$, where y_0^* is a known periodic function of t.

The solution of the first approximation equation is of the form (2.2), where $i = 0$, $R_0 = R(t, \alpha_0 u + y_0^*, \alpha_0 u^{\cdot} + y_0^{\cdot *}, 0)$, and α_0 is so far undetermined. To determine α_0 we use the basic condition of periodicity of function y_1 similar to (2.9)

$$\int_0^T R_0 \frac{u}{\Delta}\, dt = 0 \qquad (2.11)$$

Formula (2.11) can be simplified, if the conditions of periodicity of functions $y_0^{\cdot}$ and $u^{\cdot}$ are formulated as follows.

$$\int_0^T [(q_t') + (q_x')u + (q_{x^{\cdot}}')u^{\cdot} - \qquad (2.12)$$

$$(Q_{x^{\cdot 2}}'')y_0^{\cdot}u^{\cdot} - (Q_{x^2}'')y_0 u - (Q_{xx^{\cdot}}'')(y_0 u)^{\cdot}]\frac{u}{\Delta}\, dt \equiv 0$$

$$-\int_0^T [(Q''_{x^{\cdot 2}})u^{\cdot 2} + 2(Q''_{xx^{\cdot}})uu^{\cdot} + (Q_{x^2}'')u^2]\frac{u}{\Delta}dt \equiv 0 \tag{2.13}$$

Multiplying identity (2.13) by $\alpha_0^2 / 2$ and (2.12) by α_0, adding the products and subtracting their sum from (2.11) we obtain a linear equation in α_0. This equation is solvable when $P'(\tau^*) \neq 0$. We have

$$\alpha_0^* = -\frac{1}{P'(\tau^*)}\int_0^T R_0^*\frac{u}{\Delta}dt, \quad R_0^* = R(t, y_0^*, y_0^{\cdot *}, 0) \tag{2.14}$$

We also use here the identity

$$\int_0^T \frac{d}{dt}\left[(q)\frac{u}{\Delta}\right]dt = P'(\tau^*) + \int_0^T (q_t')\frac{u}{\Delta}dt \equiv 0 \tag{2.15}$$

Thus, when τ^* is a simple real root of Eq. (2.9), parameter α_0 is uniquely determined, and $y_1 = \alpha_1 u + y_1^*$, where y_1^* is a known periodic function and $\alpha_1 = \alpha_0^* + O(\varepsilon)$ is a unknown parameter.

Further derivation is by induction. Let us assume that the periodic functions y_0, $y_1, \ldots, y_{i-1}$ have been completely determined, and function $y_i = \alpha_i u + y_i^*$ has been determined to within α_i. In conformity with (2.2) parameter β_i is

$$\beta_i = -\frac{1}{T}\int_0^T dt\left\{\int_0^t [(q) + \varepsilon R_{i-1}]\frac{u}{\Delta}dt_1 - [(q) + \varepsilon R_{i-1}]\frac{v}{\Delta}\right\} \tag{2.16}$$

The unknown parameter α_i is determined by the condition of periodicity of function y_{i+1} of the kind (2.11)

$$\int_0^T R(t, \alpha_i u + y_i^*, \alpha_i u^{\cdot} + y_i^{\cdot *}, \varepsilon)\frac{u}{\Delta}dt = 0 \tag{2.17}$$

Multiplying the condition of periodicity of function $y_i^{\cdot}$ of the kind (2.12) by α_i and adding it to (2.13) multiplied by $\alpha_i^2 / 2$ we reduce Eq. (2.17) in α_i to the form

$$\alpha_i = -\frac{1}{P'(\tau^*)}\int_0^T \{[R(t, y_i^*, y_i^{\cdot *}, 0) + \tag{2.18}$$

$$r(t, \alpha_i u + y_i^*, \alpha_i u^{\cdot} + y_i^{\cdot *}, \varepsilon)]u + \varepsilon\alpha_i R_{i-1}[u^{\cdot} + (Q_{x^{\cdot}}')u]\}dt$$

Since $r(t, y, y^{\cdot}, 0) \equiv 0$, $y_i^*(t, 0) = y_0^*$, hence $\alpha_i(\varepsilon = 0) = \alpha_0^*$. For fairly small ε formula (2.18) is uniquely solvable for $\alpha_i = \alpha_i(\varepsilon)$, since its right-hand side satisfies for any fixed i the Lipschitz condition with the constant independent of subscript i, and is of order of ε. The root of Eq. (2.18) can be determined by successive approximations, and in function r it is possible to set $y = y_{i-1}$ without loss of accuracy. Parameter $\alpha_i(\varepsilon)$ is then obtained in explicit form from the linear equation.

Validity of the proposed scheme of successive approximations (2.2), (2.16), and (2.18) for deriving a periodic solution of Eq. (2.1), i. e. the fundamental $m\,/\,n$-resonance solution of (1.1), can be proved by the related procedure described in [1]. Convergence to the exact solution for fairly small $\varepsilon > 0$ is defined by the power relationship $y_{i+1} = y_i + O(\varepsilon^{i+1})$.

Theorem 2.1. When the conditions indicated Sect. 1 and 2 for periodicity and smoothness, and for a reasonably small $\varepsilon > 0$, the perturbed equation (1.1) has a simple $m\,/\,n$-resonance solution of the form $x = x_0(\psi, a^*) + \varepsilon y(t, \varepsilon)$, where y is a periodic solution of Eq. (2.1), bounded when $\varepsilon \to 0$, provided that:

1) the resonance equation (1.6) has a simple root a^* which is within the admissable range, and

2) equation (2.9) for the phase constant τ has a simple real root τ^*.

Note 2.1. The cases in which Eqs. (1.6) and (2.9) have multiple roots, are critical, and require additional consideration with the use of the general method of Poincaré [1, 6]. Solutions are constructed in the form of series or by successive approximations in fractional powers of the small parameter ε, this may be accompanied by splitting of trajectories, i. e. several perturbed trajectories may correspond to multiple roots a^* and τ^*, Supplementary conditions for the existence of a stable solution become necessary and the requirements for smoothness are increased.

Note 2.2. The case in which Eq. (2.9) is identically satisfied by τ relates to motions of higher order [1, 2, 6-8]. Sufficient conditions of their existence can be obtained by the Poincaré method similarly to Theorem 2.1. Since the direct anaylsis of system (1.1) leads to immensely cumbersome formulas, it is convenient to use the system with a rotating phase [8].

If, however, Eq. (1.6) is identically satisfied by a, i. e. $\omega = \text{const}$, a simpler "quasi-linear" system is obtained whose analysis reduces to that of a conventional system [1].

3. Investigation of stability of perturbed resonance equations. The theorem on the stability of the derived periodic solutions with constantly acting perturbations [5], no matter how small the parameter $\varepsilon > 0$, cannot be applied to system (1.1), since the generating solution is unstable, and a single group of solutions corresponds to the double zero characteristic index of the system in variations for $\varepsilon = 0$. Hence for the investigation of the perturbed motion stability it is necessary to take into account higher powers of the small parameter ε. The analysis shows that the expansion of characteristic indices are in powers $\delta = \sqrt{\varepsilon}$, i. e. that a more complex critical case is present [1-5].

The problem reduces to the analysis of the stability of the quiescence point $W = W^{\cdot} = 0$ of the variational equation

$$W^{\cdot\cdot} + Q_{x^{\cdot}}'W^{\cdot} + Q_x'W = \varepsilon\,(q_{x^{\cdot}}'W^{\cdot} + q_x'W) \tag{3.1}$$

which is obtained from (1.1) by the substitution $x = x(t, \varepsilon) + W$ and the rejection of nonlinear terms. According to investigations of Floquet-Liapunov the linear equation (3.1) with periodic coefficients has the solution $W = w \exp \lambda t$, where λ is the characteristic index and w is a periodic function of period T It satisfies the equation

$$w^{\cdot\cdot} + (Q_{x^\cdot}' + 2\lambda) w^\cdot + (Q_x' + \lambda Q_{x^\cdot}' + \lambda^2) w = \varepsilon q_{x^\cdot}' w^\cdot + \varepsilon \cdot (q_x' + \lambda q_{x^\cdot}') w \tag{3.2}$$

We have to find such λ for which Eq. (3.2) has a nontrivial periodic solution. The unknown λ and w are defined as follows :

$$\lambda = \delta\lambda_1 + \delta^2\lambda_2 + \delta^3\lambda_3(\delta), \quad w = w_0 + \delta w_1 + \delta^2 w_2 + \delta^3 w_3(t, \delta) \tag{3.3}$$

According to (3.3) the periodic functions w_0 and w_1 are defined by : $w_0 = c_0 u$ and $w_1 = c_1 u + \lambda_1 c_0 v$, where c_0 and c_1 are constants, and u and v are periodic functions defined by formulas (2.4)-(2.6). Then the equation for w_2, which is obtained by substituting (3.3) into (3.2) and equating coefficients at $\delta^2 = \varepsilon$ implies that

$$w_2 = \lambda_2 c_0 v + \lambda_1 c_1 v + c_2 u + c_0 w_2^*(t, \lambda_1^2), \qquad c_2 = \text{const}$$

Function w_2^* is periodic, if the condition of the kind (2.17), which should be considered as an equation in λ_1, is satisfied. Using methods described in Sects. 1 and 2, we reduce it to the form

$$\lambda_1^2 \int_0^T [2v^\cdot + (Q_{x^\cdot}') v + u] \frac{u}{\Delta} dt = -\int_0^T (q_t') \frac{u}{\Delta} dt$$

where the right-hand side is understood to have the same meaning as in (2.15). On the basis of definition (2.7)

$$\int_0^T [2v^\cdot + (Q_{x^\cdot}') v + u] \frac{u}{\Delta} dt = \int_0^T \left[(2v^\cdot + u)\frac{u}{\Delta} + \left(\frac{1}{\Delta}\right) uv\right] dt = T$$

and the equation for λ_1 reduces to the form

$$\lambda_1^2 = P'(\tau^*) / T \tag{3.4}$$

It follows from (3.4) that when $P' > 0$ one of the roots λ_1 is positive and the quiescence point of Eq. (3.1) is unstable, since one of the characteristic indices has necessarily a positive real part. If $P' < 0$ both roots are pure imaginary in the calculated approximation, and stability is determined by λ_2.

A more exact calculation of the characteristic index is obtained by writing the equation for the zero approximation of function w_3 as was done for w_2. The condition of w_3 periodicity determines the sought λ_2. To simplify the derived formulas it is necessary to multiply the equation for w_2 by w_1 / Δ, subtract it from the equation for w_1 multiplied by w_2 / Δ, and integrate with respect to t from 0 to T. The obtained thus combination is then added to the condition of periodicity of w_3, multiplied by the constant c_0. As a result, for λ_2 we obtain the formula

$$\lambda_2 = \frac{1}{2T} \int_0^T [(q_{x^\cdot}') - (Q_{x^{\cdot 2}}'') y_0^\cdot - (Q_{x^\cdot x}'') y_0] \, dt \tag{3.5}$$

Theorem 3.1. The perturbed m / n-resonance solution of Eq. (1.1) is stable when $\varepsilon > 0$ is reasonably small, and is so asymptotically, if
$P'(\tau^*) < 0$ (a necessary condition), and
parameter λ_2 calculated by (3.5) is negative.

Note 3.1. The quantity $P'(\tau^*) \neq 0$ by virtue of condition 2) of Theorem 2.1.

Note 3.2. When $\lambda_0 = 0$ it is necessary to calculate $\lambda(\varepsilon)$ more exactly, which implies higher requirements for the system smoothness [2].

Note 3.3. Unlike in systems close to conservative investigated in [1-3], here the expression for λ_2 contains two supplementary terms with the negative sign (see (3.5)). These quantities are determined not only by the generating solution x_0 but, also, by the zero approximation y_0 of the periodic addition $y(t, \varepsilon)$. They correspond to certain additional coefficients of "viscous friction". In the particular case of system (1.1) considered below the contribution of these additions to λ_2 is zero.

4. The particular case. Example of calculation of link mechanism resonance rotations. Let the Lagrangian function of the unperturbed system be of the form $L = \mu(x) x^{\cdot 2} / 2 - U(x)$, where $\mu(x) \geqslant \mu^\circ > 0$ is the mass which depends on the generalized coordinate x, $U(x)$ is the potential, and the work of external perturbing forces is defined by $\varepsilon \int f(t, x, x^\cdot) x^\cdot dt$, where ε is a small parameter and f is a periodic function of t. In the case of rotations all functions are assumed to be 2π-periodic in x.

The equation of motion of the perturbed system reduces to the form (1.1)

$$Q = (\mu' / \mu) x^{\cdot 2} / 2 + F, \quad F = U' / \mu, \quad q = f / \mu \tag{4.1}$$

where the prime denotes a derivative with respect to x

We consider a vibration-rotation system. When $\varepsilon = 0$ the energy integral $h = \mu(x_0) x_0^{\cdot 2} / 2 + U(x_0) = \text{const}$. Since U is a smooth periodic function, hence it attains in the interval $x \in [0, 2\pi]$ its minimum $U_1 = U(x_1)$ and maximum $U_2 = U(x_2)$ values, and $U_1 \leqslant h$. Let us assume, for simplicity, that this occurs once and that $F(x) = 0$ only at points $x = x_{1,2}$. Then for $U_1 < h < U_2$ oscillations in the system occur within the limits $x_0 \in [\xi_1, \xi_2]$, where $\xi_1(h)$ and $\xi_2(h)$ are simple roots of the equation $U(x_0) = h$. When $h > U_2$ we have either forward $(x_0^\cdot \geqslant \alpha > 0)$ or reverse $(x_0^\cdot \leqslant -\alpha < 0)$ rotations.

The periods of natural oscillations and rotations are, respectively, defined by

$$T_0(h) = 2\int_{\xi_1(h)}^{\xi_2(h)} \frac{dx}{v(x, h)}, \quad T_0(h) = \int_0^{2\pi} \frac{dx}{v(x, h)}, \quad v = \left\{\frac{\mu(x)}{2[h - U(x)]}\right\}^{1/2} \tag{4.2}$$

and the periodic motion itself is determined by formula

$$\psi = \omega(h) \int \frac{dx_0}{x_0^\cdot(x_0, h)}, \quad x_0^\cdot = \pm v, \quad \psi = \omega(h)(t + \tau) \tag{4.3}$$

Let the quality h^* be determined for some relatively prime integers m and n by the resonance condition (1.6). Then Eq. (2.9) in τ can, with allowance for (4.3), be written in the form

$$P(\tau) = \frac{\mu_0}{\Delta_0}\int_0^T f(t, x_0, x_0^{\bullet})\, x_0^{\bullet} dt = \frac{\mu_0}{\Delta_0}\oint_n f\left(\int \frac{dx}{x_0^{\bullet}(x, h^*)} - \tau, x, x_0^{\bullet}\right) dx = 0 \tag{4.4}$$

For the rotational mode formula (4.4) is simplified

$$P(\tau) = \frac{\mu_0}{\Delta_0}\int_0^{2\pi n} (f)\, dx = 0 \tag{4.5}$$

Formula (2.7) was used in (4.4) and (4.5) for the Wronskian

$$\Delta = \Delta_0 \exp \int (Q_{x^{\bullet}}') \, dt = \Delta_0 \mu_0 / \mu(x), \quad \mu_0 = \mu(x(0))$$

If τ^* is a simple real root of the transcendental equation (4.4) (or (4.5)), a resonance solution of the perturbed equation exists and is unique when $\varepsilon > 0$ is reasonably small. In particular formula (4.3) determines that root with the error $O(\varepsilon)$ for all $|t| < \infty$, when $h = h^*$ and $\tau = \tau^*$. Higher approximations are derived with the use of formula (2.2) and, if in that case $F = U' \equiv 0$, then function $z_2 = ut$ and $\Delta = u^2 = u^2(0)\,\mu_0 / \mu(x_0)$.

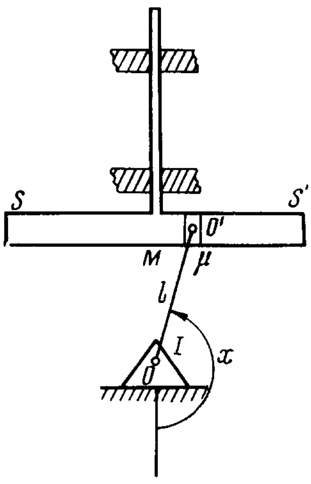

Fig. 1

For the analysis of stability it is necessary to determine functions $(Q''_{x^{\bullet 2}}) = \mu' / \mu$ and $(Q''_{x^{\bullet}x}) = (\mu' / \mu)' x_0^{\bullet}$. It is interesting that (see Note) 3.3) in the considered system the additional terms for λ_2 cancel each other. In fact

$$\int_0^T (Q''_{x^{\bullet 2}})\, y_0^{\bullet} dt = \int_0^T \left(\frac{\mu'}{\mu}\right) y_0^{\bullet} dt = y_0 \frac{\mu'}{\mu}\Big|_0^T - \int_0^T \left(\frac{\mu'}{\mu}\right)' x_0^{\bullet} y_0\, dt$$

Thus the stationary resonance solution is asymptotically stable for one of the values of τ^*, if the equivalent coefficient εf of viscous friction of external forces is negative, i.e. if [2, 3]

$$\lambda_2 = \frac{1}{2T}\int_0^T (f_x{}') \, dt < 0 \tag{4.6}$$

To illustrate the method we investigate below an example of the basic ($m = n = 1$) resonance rotations of symmetric reciprocating motion of a slide-and-crank linkage [9] lying in a plane normal to the force of gravity. It is shown in plan view in Fig. 1, where M denotes the mass of the slideway SS', the mass of the slider O', I the moment of inertia of the crank OO' relative to the center of rotation l the crank length, and x denotes the angular velocity.

In the considered problem $U' \equiv 0$ and the system energy is

$$h = [I + l^2 (\mu + M \sin^2 x)] \, x^{\cdot 2} / 2 = \text{const} > 0 \tag{4.7}$$

The unperturbed rotational motion x_0 and its period T_0 are defined by the first integral in (4.7)

$$\psi = \frac{\pi}{2}\left[1 - \frac{E(\pi/2 - x_0, k)}{\mathbf{E}(k)}\right], \quad T_0(h) = 2l\sqrt{\frac{2M}{h}} \, \frac{\mathbf{E}(k)}{k} \tag{4.8}$$

$$k = l\,\{M / [I + l^2(\mu + M)]\}^{1/2} \quad (0 < k < 1)$$

where $E(\varphi, k)$ is an elliptic integral of the second kind and $\mathbf{E}(k)$ is a complete elliptic integral of the second kind.

We assume that a small periodic force $\varepsilon f_0 \sin(\nu t + \gamma)$ parallel to the guideways SS' acts on the slider O', while the slideway exerts on it the small force of viscous friction proportional to the relative velocity $-2\varepsilon l \lambda \, x^{\cdot} \cos x$ ($\lambda > 0$). The moment of perturbing forces is then

$$\varepsilon f = \varepsilon f_0 \, l \cos x \sin(\nu t + \gamma) - 2\varepsilon\lambda l^2 x^{\cdot} \cos^2 x$$

where $\varepsilon \geqslant 0$ is a small numerical parameter and f_0, ν, γ, and λ are constants.

Equations of the kind (4.5) that define the phase constant τ in (4.8) reduce to the form

$$R\cos(\nu\tau + \gamma + \theta) - C = 0 \tag{4.9}$$

$$R = \sqrt{A^2 + B^2}, \qquad \cos\theta = A / R, \ \sin\theta = B / R$$

$$A = \frac{f_0 l}{\Delta_0}\int_0^{2\pi} \cos x \cos \frac{E(\pi/2 - x, k)}{\mathbf{E}(k)} \, dx,$$

$$B = \frac{f_0 l}{\Delta_0}\int_0^{2\pi} \cos x \sin \frac{E(\pi/2 - x, k)}{\mathbf{E}(k)} \, dx$$

$$C = 8\frac{\lambda l^2}{\Delta_0 k}\sqrt{\frac{2h^*}{M}} \, [\mathbf{K}(k) - \mathbf{E}(k)]$$

$$h^* = 2M\left(\frac{l\nu}{\pi}\right)^2 \frac{\mathbf{E}^2(k)}{k^2}, \qquad \Delta_0 = x_0^{\cdot 2}(0) = \frac{2h^*}{I + \mu l^2} > 0$$

where $\mathbf{K}(k)$ is a complete elliptic integral of the first kind. Formulas for coefficients A, B, and C can be approximately calculated by the powers of modulus k with any desired degree of accuracy. Such calculations are effective for small k, when the unperturbed motion is close to uniform rotation at velocity $[2h^* / (I + \mu l^2)]^{1/2}$; in particular

$$A = O(k^4), \quad B = (f_0 l / \Delta_0)(\pi - k^2/8) + O(k^4)$$

$$C = \pi (2\lambda l^2 / \Delta_0)(1 + {}^3/_8 k^2)[2h^* / (I + \mu l^2)]^{1/2} + O(k^4)$$

When $|C / R| < 1$ Eq. (4.9) has in the interval of 2π length two simple real roots

$$\tau_{1,2} = (1/\nu)[\pm \arccos (C/R) - \theta - \gamma]$$

To each of roots $\tau_{1,2}$ corresponds one stationary solution, when $\varepsilon > 0$ is reasonably small. Since according to (4.8) the dependence of x_0 on h is expressed in terms of the phase, hence $z_2 = ut$, which implies that $\Delta = u^2$, and the addition εy to x_0 is obtained with the use of the simpler scheme

$$y_{i+1} = \alpha_{i+1} u + u \left\{ \int_0^t dt_1 \int_0^{t_1} [(q) + \varepsilon R_i] \frac{dt_2}{u} - \frac{1}{T_0} \int_0^T dt \int_0^t [(q) + \varepsilon R_i] \frac{dt_1}{u} \right\}$$

$$\int_0^T R_i \frac{dt}{u} = 0, \quad i = 0, 1, \ldots$$

$$(q) + \varepsilon R_i = f_i \{I + l^2 [\mu + M \sin^2 (x_0 + \varepsilon y_i)]\}^{-1}, \quad f_i = f(t, \ x_0 + \varepsilon y_i, x_0^{\cdot} + \varepsilon y_i^{\cdot})$$

For one of roots $\tau_{1,2}$ the quantity $P'(\tau) < 0$, consequently the perturbed rotation that corresponds to that root is asymptotically stable, since $\lambda_2 = -\lambda < 0$.

If the system is in a uniform gravitational field, the second integral of unperturbed motion (4.8) and the periods of oscillations or rotations (4.2) are expressed in terms of elliptic integrals of the third kind. Let the mechanism be in such position that the motion of the slideway occurs along the gravity force vector. Then

$$U(x) = [(\mu + M) l + md] g (1 - \cos x)$$

where m is the mass of the crank and d its "arm". Let the system be subjected to the moment of external periodic forces and to the friction force

$$f = f_0 l \cos x \sin (\nu t + \gamma) - 2\lambda l^2 x^{\cdot} \cos^2 x$$

and let

$$g[(\mu + M) l + md] / \nu^2 (I + \mu l^2) \sim \varepsilon$$

i.e. the problem is that of rapid forces rotation. Then, if we introduce the "brief time" $s = \nu t$ and assume that the ratios $f_0 l / (I + \mu l^2) \nu^2$ and $2\lambda l^2 / (I + \mu l^2) \nu$ are small quantities of the same order ε, formulas (4.7)-(4.9) remain unchanged, because the integral of the kind (4.5) of the perturbed potential force

$$\int_0^{2\pi} \frac{\sin x_0 u \, ds}{[I + l^2 (\mu + M \sin^2 x_0)] \Delta} = \frac{\mu_0}{\Delta_0} \int_0^{2\pi} \sin x \, dx = 0$$

remains unchanged.

Thus the problem of the slide-and-crank mechanism rapid rotation reduces to that investigated previously.

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